PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
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THE ROYAL SOCIETY

Efficiency concepts in capital accumulation models

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This paper discusses alternative definitions of efficiency and their relationship. The mean-variance approach is contrasted with a growth-security approach based on quantiles of the wealth distribution. The approaches are equivalent when the asset returns follow geometric Brownian motion. However, the quantiles approach offers significant advantages when lognormality does not hold, as with most financial assets.

Keywords: efficiency; dominance; mean variance; capital growth

1. Introduction

Efficiency of performance is an important concept for the assessment of financial systems. In general terms efficiency implies achieving output performance standards with a minimum of effort. The measurement of output and effort in a dynamic system involves distance and time. For example, with competing strategies for achieving a goal, the one which requires less distance is more efficient. Alternatively, the strategy which achieves the goal in less time is considered more efficient. The direction of movement of objects in actual systems has a degree of uncertainty, as does the path to a goal under a given strategy. The ordering of strategies that have many possible performance trajectories requires the notion of stochastic dominance. That is, the set of possible performance paths for one strategy can be compared with the set of paths for an alternative strategy with a dominance criterion. Dominance could be exhibited in the distance or time dimension.

In capital accumulation under uncertainty, dominance and efficiency are fundamental. The general concept of dominance is defined through expected utility (Hanoch & Levy 1969). An equivalent formulation has been developed using probability distributions (Blackwell 1951; Rothschild & Stiglitz 1970). The general criteria are difficult to implement, and have been relaxed to criteria defined by moments of the probability distribution for accumulated capital. In particular, the use of means and variances to order strategies and characterize efficiency has been widespread (Markowitz 1959, 1987; Samuelson 1970). An alternative approach to relaxing the general criteria is to focus on quantiles of the probability distributions, presumably at significant levels of accumulated capital (Roy 1952; Pyle & Turnovsky 1970; MacLean & Ziemba 1991; MacLean *et al.* 1999).

This paper considers alternative definitions of efficiency and their relationship. In § 2 the capital accumulation process is developed. Definitions of dominance and efficiency are presented in § 3. A comparison of definitions and associated efficient sets is in § 4.

2. Capital accumulation under uncertainty

The framework for wealth accumulation is continuous time investment in a risky asset and a risk free asset. If $P(t)$ is the price of the risky asset traded at time t, then the dynamics of the instantaneous return on the asset is given by the stochastic differential equation

$$\frac{\mathrm{d}P}{P} = \alpha \,\mathrm{d}t + \sigma \,\mathrm{d}z, \tag{2.1}$$

where α is the instantaneous expected rate of return, σ is the standard deviation of the rate of return and $\mathrm{d}z$ is standard Brownian motion. For the risk-free asset the instantaneous return at time t is known to be $r(t)$.

The market defined by the risky and risk-free assets is assumed to satisfy the conditions given in Merton (1990).

(A1) Assets have limited liability.

(A2) There are no transactions costs, taxes, and assets are infinitely divisible.

(A3) The capital market is competitive, always in equilibrium and investors are price takers.

(A4) Capital can be borrowed or lent without limit at the rate r.

(A5) Short sales of assets are allowed.

(A6) Trading takes place continuously in time.

At time t, with wealth W_t, the investor must decide the fraction of wealth x to invest in the risky asset, with $1-x$ held in the risk-free asset. The x is unconstrained since the risk-free fraction can be varied through borrowing or lending to meet the budget constraint. It is assumed that x, a fixed mix strategy, is independent of t, as are the parameters α and σ. (In practice these parameters are estimated from information asset prices. The investment fraction is conditional on the most recent estimates and in that sense is dynamic, e.g. Browne & Whitt (1996).)

The dynamics of wealth (accumulated capital) through investment at time t are given by

$$\mathrm{d}W = (x(\alpha - r) + r)W \,\mathrm{d}t + xW\sigma \,\mathrm{d}z. \tag{2.2}$$

The wealth process (2.2) is a geometric Brownian motion and correspondingly $\ln W$ is arithmetic Brownian motion. A representation of a future wealth path resulting from a decision to invest xw_0 of starting wealth w_0 in the risky asset is shown in figure 1. Also shown is the log-wealth at a time t units in the future and the *time* t^* when wealth first reaches the level w^*.

For the Brownian motion of log-wealth, the distribution for wealth at a fixed point in time and the distribution for first passage time to a fixed wealth are known (Cox & Miller 1970). Consider the notation $\mu(x) = x(\alpha - r)$ and $\sigma(x) = x\sigma$. Let $W_t(x, w_0)$

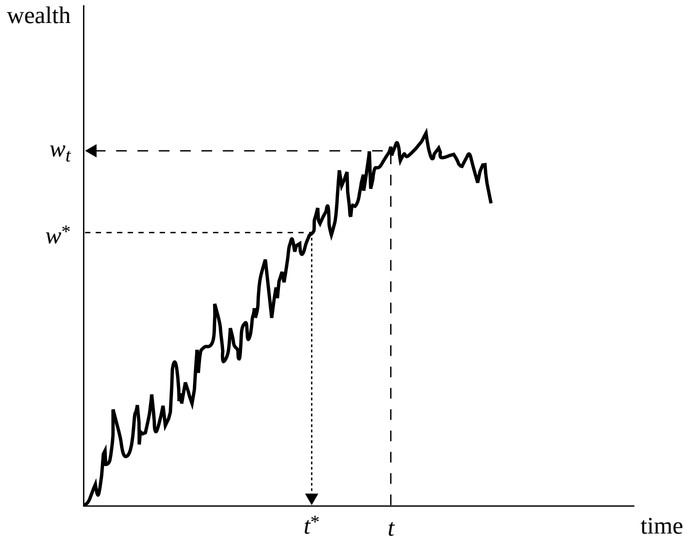

Figure 1. Wealth trajectory.

denote the wealth at time t, starting at w_0, from investing the fraction x in the risky asset, and $T_{w^*}(x, w_0)$ denote the first passage time to wealth w^*, starting at w_0, from investing the fraction x in the risky asset.

Then W_t has distribution function F_t, and T_{w^*} has distribution function G_{w^*}, where the corresponding densities are

$$f_t(w; x \mid w_0) = \frac{1}{\sqrt{2\pi t\sigma^2(x)}} \exp\left[\frac{-(\ln(w/w_0) - \frac{1}{2}D(x)t)^2}{2t\sigma^2(x)}\right], \tag{2.3}$$

$$g_{w^*}(t; x \mid w_0) = \frac{\ln(w^*/w_0)}{\sqrt{2\pi t^3\sigma^2(x)}} \exp\left[\frac{-(\ln(w^*/w_0) - \frac{1}{2}D(x)t)^2}{2t\sigma^2(x)}\right], \tag{2.4}$$

and $D(x) = 2\mu(x) - \sigma^2(x)$. The distributions F_t and G_{w^*} are for the lognormal and inverse normal, respectively.

From the distributions for wealth and time the characteristics relevant for later discussion can be calculated.

(i) *Wealth*

$$\text{mean:} \qquad E(W(t) \mid x, w_0) = w_0 \exp(\tfrac{1}{2}D(x)t), \tag{2.5}$$

$$\text{variance:} \qquad V(\ln W(t) \mid x, w_0) = \sigma^2(x), \tag{2.6}$$

$$\text{quantile:} \quad \Pr[W(t) \geqslant m_\alpha \mid x, w_0] = \Phi\left(\frac{\frac{1}{2}D(x)t - \ln(m/w_0)}{\sigma(x)\sqrt{t}}\right) = \alpha, \tag{2.7}$$

where Φ is the standard normal cumulative distribution function. (For reference on the lognormal distribution, see, for example, Kendall & Stuart (1969), pp. 168–169.)

(ii) *Time*

In considering the first passage time to wealth goals, there are two levels (quantiles) of interest: an upper growth value M and a lower decay value m with $m < w_0 < M$.

$$\text{mean:} \qquad E(T_M \mid x, w_0) = \frac{\ln(M/w_0)}{\frac{1}{2}D(x)}, \tag{2.8}$$

$$\text{variance:} \qquad V(T_M \mid x, w_0) = \frac{\ln(M/w_0)\sigma^2(x)}{\frac{1}{4}D^3(x)}, \tag{2.9}$$

$$\text{quantile:} \quad \Pr[T_M < T_m \mid x, w_0] = \frac{1-(m/w_0)^{2(\mu(x)/\sigma^2(x))-1}}{1-(m/M)^{2(\mu(x)/\sigma^2(x))-1}} = \alpha. \tag{2.10}$$

(For reference on the inverse normal distribution, see, for example, Cox & Miller (1970), pp. 220–222.)

3. Dominance and efficiency

The accumulated capital up to time t, $W_t(x)$, and the first passage time to goal M, $T_M(x)$, are random variables with distributions depending on the investment strategy x = fraction of wealth allocated to the risky asset. The ordering of distributions and thereby the ordering of strategies requires a dominance relation. The standard definition, generally attributed to Lehman (1959), is given in Hanoch & Levy (1969).

Definition 3.1. $W_t(x_1)$ dominates $W_t(x_2)$ if $Eu(W_t(x_1)) \geqslant Eu(W_t(x_2))$ for every monotone utility function u, with strict inequality for some u.

An equivalent statement of first-order stochastic dominance in terms of distributions is as follows.

Definition 3.2. $W_t(x_1)$ dominates $W_t(x_2)$ if and only if $F_t(w; x_1) \leqslant F_t(w; x_2)$ for every w, and $F_t(w_1; x_1) < F_t(w_1; x_2)$ for some w_1.

Obviously, the dominance criteria apply to the random first passage times $T_M(x_1)$, $T_M(x_2)$, although it is not clear that the ordering corresponds to the wealth results. For the class of power utility functions $u_\alpha(w) = (1/\alpha)w^\alpha$, $\alpha < 1$, a type of equivalence holds. In particular, maximizing the expected utility of end-of-horizon wealth yields the same strategy as minimizing the expected time to the appropriate wealth goal. Alternatively, the dynamics of log-wealth can be used to link the wealth and time distributions. Consider $W_t(x)$, the wealth at time t for investment strategy x. For each value $W_t(x) = w_t$, the time to the goal M can be derived from the Brownian motion $\mathrm{d}(\log W) = \mu(x)\,\mathrm{d}t + \sigma(x)\,\mathrm{d}z$. Define the conditional random variable $T_M(x \mid w_t)$ as the time to M from w_t at time t under strategy x. For $w_t > M$ the process works backward in time at the rate $\mu(x)$. If $G_M(t; x \mid w_t)$ is the distribution for $T_M(x \mid w_t)$, then $G_M(t; x) = E_{w_t}\{G_M(t; x \mid w_t)\}$, where E_{w_t} denotes expectation with respect to the distribution for $W_t(x)$. Furthermore, it follows that $F_t(w; x_1) \leqslant F_t(w; x_2)$, for every w with strict inequality for some w_1, implies $G_M(t; x_1) \geqslant G_M(t; x_2)$, with strict inequality for some t.

Reversing the roles of wealth and time yields the other direction and it is concluded that

$$W_t(x) \text{ dominates } W_t(x_2) \Leftrightarrow T_M(x_1) \text{ dominates } T_M(x_2).$$

It is possible to consider efficiency in either the wealth or time dimension with comparable results. Personal preference would dictate whether it is easier to work with a time horizon or wealth goal.

In practical terms it is difficult to work with a criterion for dominance based on all utility functions or distribution functions. If distributions are summarized by a small number (vector) of measures, then comparisons are based on a partial ordering of vectors. There are two common approaches to distribution summary: (i) moments; and (ii) quantiles. These are the basis of *relaxed* definitions of dominance. With vectors $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{u} = (u_1, u_2)$, denote $\boldsymbol{v} < \boldsymbol{u}$ if and only if $v_i \leqslant u_i$, with strict inequality in one component.

Definition 3.3. $W_t(x_1)$ mean variance dominates $W_t(x_2)$ if and only if

$$(-E\{\ln W_t(x_1)\}, V\{\ln W_t(x_1)\}) < (-E\{\ln W_t(x_2)\}, V\{\ln W_t(x_2)\}).$$

Alternatively, $T_M(x_1)$ mean variance dominates $T_M(x_2)$ if and only if

$$(E\{T_M(x_1)\}, V\{T_M(x_1)\}) < (E\{T_M(x_2)\}, V\{T_M(x_2)\}).$$

The mean-variance criterion applied to wealth has been studied extensively. For the continuous-time model considered here, with lognormally distributed wealth, the mean-variance criterion is equivalent to the more general definition. For other distributions, the restriction to the first two moments could produce results which differ from the general criterion.

The quantile approach to distribution summary is robust; that is, it does not depend on the functional form of the distribution. However, considering a few points on a distribution for comparison may lead to an incorrect conclusion.

Consider the quantile m_α such that $\Pr[W(t) \geqslant m_\alpha \mid x, w_0] = \alpha$. Then let $Q_t(x) = m_{0.5}$ denote the median log-wealth with strategy x, and $S_t^m(x) = \Pr[W_t(x) \geqslant m]$ denote the chance of exceeding subsistence with strategy x, for subsistence level m. In the time dimension, $Q_M(x_1)$ denotes the median time to M and $S_M^m(x) = \Pr[T_M(x) < T_m(x)]$ denotes the chance of reaching goal M before falling to subsistence m.

Definition 3.4. $W_t(x_1)$ growth security dominates $W_t(x_2)$ if and only if

$$(Q_t(x_1), S_t^m(x_1)) > (Q_t(x_2), S_t^m(x_2)).$$

Alternatively, $T_M(x_1)$ growth security dominates $T_M(x_2)$ if and only if

$$(Q_M(x_1), S_M^m(x_1)) > (Q_M(x_2), S_M^m(x_2)).$$

For the Brownian motion model and lognormal wealth, the quantiles are functions of the moments (mean, variance) as shown in formulae (2.5)–(2.10). For this special case, the dominance criteria are consistent. MacLean *et al*. (1992) developed formulae for the moments and quantiles for general distributions in discrete time (see also MacLean & Ziemba 1999). In that situation the moment and quantile relations are not the same. It can be argued that the non-parametric quantile representation is a better approximation to the distribution-based dominance.

The relaxed-dominance criteria based on mean variance or growth security lead to alternative notions of efficiency. An investment strategy x is *mean-variance efficient* if its corresponding wealth $W_t(x)$ (or time to goal $T_M(x)$) is *undominated* by the mean-variance criterion. Similarly, *growth-security efficiency* implies that wealth or time is undominated by the growth-security criterion.

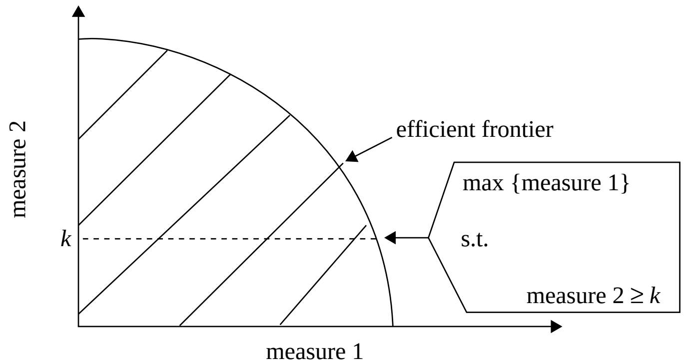

Figure 2. Investment opportunity set.

Table 1. *Efficient frontier problems*

	mean variance	growth security
log-wealth	$\max_x D(x)$ s.t. $\sigma^2(x) \leqslant \sigma_0^2$	$\max_x D(x)$ s.t. $\Pr[W_t(x) \geqslant m] \geqslant \alpha_0$
time	$\min_x \dfrac{1}{D(x)}$ s.t. $\dfrac{\sigma^2(x)}{D^3(x)} \leqslant V_0$	$\min_x \dfrac{1}{D(x)}$ s.t. $\Pr[T_M(x) < T_m(x)] \geqslant \alpha_0$

4. Efficient sets

The definitions of dominance and efficiency produce sets of investment strategies which are 'optimal' in the sense that they cannot be dominated. These efficient strategies are on the boundary or frontier of the opportunity set of performance measures for feasible strategies. A general representation of the opportunity set and efficient frontier is shown in figure 2.

If attention is focused on the mean-variance and growth-security efficiency, then the relevant performance measures have explicit formulae, which are given in equations (2.5)–(2.10) for the Brownian motion model. Therefore, the efficient frontier can be found from the simple optimization problems displayed in table 1.

In table 1 the objective and constraint functions could be reversed, and that is often the case in mean-variance formulations. The concept of minimizing variance in an efficient solution is traditional, particularly in statistics. The equivalent representation with mean objective is familiar in expected utility and gives each problem the same objective. The growth-security (quantile) objective is to maximize the median, but in the lognormal case this is equivalent to the mean log-wealth.

There are a number of results about the efficient frontier for each model which will be established.

(R1) In each case the unconstrained optimum is $x^* = (\alpha - r)/\sigma^2$. The fully constrained (uniquely feasible) optimum is $x = 0$. We call x^* the optimal-growth and 0 the optimal-security strategy, respectively.

(R2) Any efficient strategy can be written as a linear combination of the optimal-growth and optimal-security strategies. As the security constraint is relaxed, the efficient frontier is traced out.

(R3) In the range $0 \leqslant x \leqslant x^*$, the variance/security measures in the constraints are monotonic, and the optimum can be found by inverting the constraint relation.

(R4) If wealth rather than log-wealth is considered, then there exist mean-variance efficient strategies $x > x^*$. However, these strategies are not efficient by the growth-security or the general-dominance criteria.

Proof. In each case the unconstrained optimum is

$$x^* = \arg\max_x\{D(x)\} = \arg\max\{2[x(\alpha - r) + r] - x^2\sigma^2\} = \frac{\alpha - r}{\sigma^2}.$$

The fully constrained case is with $\sigma_0^2 = V_0 = 0$ (or $\alpha_0 = 1$), where everything is invested in the risk-free asset.

With the investments in the risk-free and risky funds $(1 - x, x)$, and the optimal-growth strategy $(1 - x^*, x^*)$ and optimal-security strategy $(1, 0)$, then

$$(1 - x, x) = \lambda(1 - x^*, x^*) + (1 - \lambda)(1, 0),$$

where $\lambda = x/x^*$.

For variance, consider that

$$\frac{\mathrm{d}}{\mathrm{d}x}\sigma^2(x) = 2x\sigma^2 > 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sigma^2(x)}{D^3(x)}\right) = 2x\sigma^2 D^{-4}(x)[2\sigma^2x^2 - (\alpha - r)x + 2r] > 0,$$

provided the squared Sharpe ratio $(\alpha - r)^2/\sigma^2$ is less than $16r$ (that is, expected returns are not excessive).

For quantiles, consider

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi\left(\frac{\frac{1}{2}D(x)t - \ln(m/w_0)}{\sigma(x)\sqrt{t}}\right) < 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1 - k^{-E(x)}}{1 - k^{-(\delta+1)E(x)}}\right) = \frac{\mathrm{d}E}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{1 - k^{-E}}{1 - k^{-(\delta+1)E}}\right) < 0$$

for $x < x^*$, where

$$E(x) = \frac{2\mu(x)}{\sigma^2(x)} - 1, \quad k = \frac{m}{w_0} \quad \text{and} \quad k^{-(\delta+1)} = \frac{m}{M}.$$

Also,

$$\frac{\mathrm{d}}{\mathrm{d}x}D(x) > 0 \text{ for } x < x^* \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}D(x) < 0 \text{ for } x > x^*.$$

However,

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu(x) = \alpha - r > 0 \text{ for all } x.$$

For log-wealth and time the mean-variance and growth-security efficient strategies are $x \in [0, x^*]$. If the wealth perspective applies, the mean-variance efficient strategies

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are $x \in [0, \infty]$. However, from the distribution for wealth (2.3), $W_t(x^*)$ dominates $W_t(x)$ for $x > x^*$ (by criterion 3.2). Since x^* is feasible (satisfies the variance constraint), it is clear that mean-variance efficiency for wealth is not equivalent to the stochastic dominance criterion. Hanoch & Levy (1969) discuss cases when they are equivalent, such as normality of returns and quadratic utility functions. ■

5. Discussion

This paper discusses a variety of efficiency concepts applied to portfolio selection. In the idealized lognormal world, as characterized by the geometric Brownian motion for asset returns, the various approaches are equivalent. (The exception is wealth mean-variance efficiency with levered strategies beyond the optimal growth level.) It is reassuring that methods are equivalent where they should be, and this allows the investor the freedom to use a method that more readily incorporates those performance measures which are meaningful. The authors argue that measures based on wealth goals are easier to understand.

There is more to the choice of method than convenience. Empirical evidence indicates that asset returns are not lognormal (see, for example, Jackwerth & Rubinstein 1996; Mandelbrot 1997). Rather, the tails of the distribution for most financial assets are far heavier than lognormal, particularly on the downside. In this case, the use of means and variances, measures which are sensitive to outliers, would lead to more aggressive positions; that is, investments whose risk is higher. Measures which are robust or insensitive to outliers would provide strategies with risk in line with investor preferences. It is in this real world scenario where the growth-security methods, which are based on quantiles, should be most appealing.

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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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